Spectral Theorem for Bounded Self-Adjoint Operators with Projection-Valued Measures

Samuel Perales

December 2020

1 Introduction

In Functional Analysis we care about different types of linear operators $T : X \to Y$ where X and Y are Normed Linear Spaces. The purpose of Spectral Theory is to extend the ideas of eigenvalues and eigenvectors from matrices to these linear operators. In Quantum Mechanics, this problem becomes interesting as we consider X and Y to be infinite dimensional Hilbert Spaces $\mathbb{H}$ (often taken to just be $L^2([0,1])$), where true eigenvectors don't always exist for a given operator. As an example, consider the operator A defined by:

$$\begin{gathered} A : L^2([0,1]) \to L^2([0,1]) \\ (A\psi)(x) \mapsto x\psi(x) \end{gathered} \tag{1}$$

A is self-adjoint since $\langle A\phi, \psi\rangle = \langle \phi, A\psi\rangle$, but A has no non-trivial eigenvectors in $L^2([0,1])$. If we were to carry the idea of $n \times n$ self-adjoint matrices over to operators however, we would expect A to have infinitely many eigenvalue, eigenvector pairs. For this reason, we extend the idea of eigenvectors to generalized eigenvectors, which fulfill the eigenvalue equation $A\psi = \lambda\psi$ but lie in a space outside of $\mathbb{H}$.

2 Resolvent and Spectrum of an Operator

Definition 2.1: For a bounded operator A, the **resolvent** $\rho(A)$ is the set of all $\lambda \in \mathbb{C}$ such that the bounded operator $A_\lambda = (A - \lambda I)$ has a bounded inverse.

Definition 2.2: The **spectrum**, $\sigma(A)$, is the complement of $\rho(A)$ in $\mathbb{C}$.

The spectrum is often broken into disjoint pieces called the point spectrum σ_p, the continuous spectrum σ_c, and the residual spectrum σ_r depending on the properties of A_λ for λ in a given spectrum, however, we will not care about this distinction.

Lemma 2.3: If A is bounded and self-adjoint, then for all $\lambda = a + bi \in \mathbb{C}$, $\langle (A_\lambda\psi, A_\lambda\psi\rangle \geq b^2\langle\psi,\psi\rangle$.

Proof.

$$\begin{aligned} \langle (A_\lambda\psi, A_\lambda\psi\rangle &= \langle ((A-\lambda I)\psi, (A-\lambda I)\psi\rangle \\ &= \langle ((A-(a+bi)I)\psi, (A-(a+bi)I)\psi\rangle \\ &= \langle ((A-aI)\psi, (A-aI)\psi\rangle + ib\langle (\psi, (A-aI)\psi\rangle - ib\langle ((A-aI)\psi, \psi\rangle + b^2\langle\psi,\psi\rangle \\ &\geq b^2\langle\psi,\psi\rangle \end{aligned}$$

■

Proposition 2.4: If A is bounded and self-adjoint, then $\sigma(A) \subset \mathbb{R}$ and $\lambda \in \mathbb{R}$ if and only if there exists some sequence $\psi_n \in \mathbb{H}$ of nonzero vectors such that

$$\lim_{n\to\infty} \frac{\|A\psi_n - \lambda\psi_n\|}{\|\psi_n\|} = 0 \tag{2}$$

Proof. Suppose $\lambda = a + bi$ and $b \neq 0$. From Lemma 2.3, we know that $\langle(A_\lambda\psi, A_\lambda\psi\rangle \geq b^2\langle\psi,\psi\rangle > 0$ which implies A_λ is injective. This is because $ker(A_\lambda) = \{0\}$ and the operator is linear. Using the same lemma, we can notice that $A_{\bar{\lambda}}$ is also injective because it has a nonzero imaginary part. We also have that $Range(A_\lambda)^\perp = ker(A_{\bar{\lambda}}) = \{0\}$ which implies $Range(A_\lambda)$ is dense in $\mathbb{H}$. Consider a $\phi \in \mathbb{H}$ and take $\{\psi_n\} \subset Range(A_\lambda)$ such that $\phi_n = A_\lambda\psi_n$ and $\phi_n \longrightarrow \phi$. Since $\|\psi_n - \psi_m\| \leq b^{-1}\|A_\lambda(\psi_n - \psi_m)\|, \psi_n$ is Cauchy. $\mathbb{H}$ is complete so $\psi_n \longrightarrow \psi \in \mathbb{H}$.

$$A_\lambda\psi = \lim_{n\to\infty} A_\lambda\psi_n = \lim_{n\to\infty} \phi_n = \phi$$

so we have $Range(A_\lambda) = \mathbb{H}$ meaning A_λ is bijective and is boundedly invertible. For λ with this property, $\lambda \in \rho(A)$ which implies that $\lambda \notin \sigma(A)$. We conclude that all spectrum values of A are real.

Next, suppose there is some $\psi_n \in \mathbb{H}$ for which (2) holds and assume that A_λ is invertible. Then for $\phi_n = A_\lambda\psi_n$, we have $\psi_n = A_\lambda^{-1}\phi_n$ which gives

$$\lim_{n\to\infty} \frac{\|A\psi_n - \lambda\psi_n\|}{\|\psi_n\|} = \lim_{n\to\infty} \frac{\|A_\lambda\psi_n\|}{\|\psi_n\|} = \lim_{n\to\infty} \frac{\|\phi_n\|}{\|A_\lambda^{-1}\phi_n\|} = 0$$

implying A_λ^{-1} is unbounded meaning $\lambda \in \sigma(A) \subset \mathbb{R}$. For the converse, if for $\lambda \in \mathbb{R}$, there is not such a sequence, we know there must be some $\epsilon > 0$ such that $\|A_\lambda\psi\| \geq \epsilon\|\psi\|$ for any $\psi \in \mathbb{H}$. By the same argument as before, A_λ is injective and we know that $Range(A_\lambda)$ is dense in $\mathbb{H}$ and must have an inverse implying equation (2) must hold. ■

3 Projection-Valued Measures

We now know a good amount about the spectrum of a bounded self-adjoint operator. Following some definitions, we will be able to make a very strong statement regarding how we can decompose such operators.

Given some Borel set $E \subset \sigma(A)$, which by Proposition 2.4 we know is entirely real, we want an idea of a **spectral subspace** V_E as the closed span of generalized eigenvectors for A with associated eigenvalues $\lambda \in E$.

Intuitively, we can discern some properties that V_E should have:

- $V_{\sigma(A)} = \mathbb{H}$, since the span of all eigenvectors in the space should form a basis.
- $V_\emptyset = \{0\}$, which is typical for the span of an empty set of vectors.
- If $E \cap F = \emptyset$, then $V_E \perp V_F$, because we expect distinct eigenvectors to be orthogonal with one another.
- $V_{E\cap F} = V_E \cap V_F$
- V_E is invariant under A, since V_E consists of eigenvectors of A.
- If $E \subset [\lambda_0 - \epsilon, \lambda_0 + \epsilon]$ and $\psi \in V_E$, then $\|A_{\lambda_0}\psi\| \leq \epsilon\|\psi\|$, which we include because our subspaces may include generalized eigenvectors instead of true eigenvectors.

Definition 3.2: For a set X and $\sigma-$algebra $\Omega \subset X$, we call the map $\mu : \Omega \to B(\mathbb{H})$ a **projection-valued measure** if it satisfies the following properties:

- $\mu(E)$ is an orthogonal projection for all Borel sets $E \in \Omega$
- $\mu(\emptyset) = 0$ and $\mu(X) = I$
- If $E_1, E_2, E_3, ...$ in Ω are disjoint, then for all $v \in \mathbb{H}$,

$$\mu(\bigcup_{i=1}^{\infty} E_i)v = \sum_{i=1}^{\infty} \mu(E_i)v$$

This idea of projection-valued measures allows us to consider subsets E of our spectrum and find a projection onto the generalized eigenvectors in the associated subspace V_E. Such an object allows us to break our space into pieces and integrate over them to obtain a new re-weighted operator. Before doing this, we need to notice that for a projection-valued measure μ and $\psi \in \mathbb{H}$, a real valued measure $\mu_\psi(E) = \langle \psi, \mu(E)\psi \rangle$ can be defined for all $E \in \Omega$. This allows us to associate each projection-valued measure with an ordinary measure to integrate over.

Proposition 3.3 If Q is a bounded quadratic form on $\mathbb{H}$, there is a unique bounded linear operator A on $\mathbb{H}$ such that $Q(\psi) = \langle \psi, A\psi \rangle$ for all $\psi \in \mathbb{H}$. If $Q(\psi)$ belongs in $\mathbb{R}$ for all $\psi \in \mathbb{H}$, then A is self-adjoint.

Proof. A proof is given in the appendix A. 63 of [1].

Proposition 3.4 For a set X, $\sigma-$algebra $\Omega \subset X$, and projection-valued measure $\mu : \Omega \to B(\mathbb{H})$, we can uniquely associate bounded, measurable, complex-valued functions f with some operator $\int_\Omega f d\mu$ such that

$$\langle \psi, (\int_X f d\mu)\psi \rangle = \int_X f d\mu_\psi \tag{3}$$

Such an integral must also have the following:

- $\int_X 1_E d\mu = \mu(E)$, where 1_E is the characteristic function of E. That is, the integral over some subset E will give us the projection onto its associated spectral subspace V_E.
- For all bounded, measurable f on Ω,

$$\left\| \int_X f d\mu \right\| \leq \sup_{\lambda \in X} |f(\lambda)| \tag{4}$$

- For all bounded, measurable f and g on Ω,

$$\int_X fg d\mu = (\int_X f d\mu)(\int_X g d\mu) \tag{5}$$

- For all bounded, measurable f on Ω,

$$\int_X \bar{f} d\mu = (\int_X f d\mu)^* \tag{6}$$

Proof. Given a projection-valued measure μ and bounded, measurable function f, we define $Q_f : \mathbb{H} \to \mathbb{C}$ by

$$Q_f(\psi) = \int_X f d\mu_\psi$$

which for characteristic functions 1_E, we know is a bounded quadratic form. By Proposition 3.3, we have that $Q_f(\psi) = \langle \psi, A_f \psi \rangle$ for some unique bounded operator A_f and all $\psi \in \mathbb{H}$. If we define $A_f = \int_X f d\mu$, we satisfy (3) and give uniqueness to the map $f \mapsto \int_X f d\mu$ by the same proposition. Since Q_f is bounded quadratic for characteristic functions it will also be bounded quadratic for any bounded measurable f in general. We can also see that

$$|Q_f(\psi)| \leq (\sup_{\lambda \in X} |f(\lambda)|) \|\psi\|^2$$

which can be used to show (4). For (5), we have multiplicativity of the integrals at a characteristic function level and can expand on this idea to show the property for all bounded measurable f and g. Lastly, we notice that for real-valued f, $Q_f(\psi)$ will be real valued implying A_f is self-adjoint by Proposition 3.3. Further details of this proof are given in [1].

■

4 Spectral Theorem

Theorem 4.1 (Spectral Theorem for Bounded Self-Adjoint Operators) For a bounded, linear, self-adjoint operator A on $\mathbb{H}$, there exists a unique projection-valued measure μ^A on the Borel $\sigma-$algebra Ω for $\sigma(A)$ such that

$$\int_{\sigma(A)} \lambda d\mu^A(\lambda) = A \tag{7}$$

Although a proof of this statement will not be given explicitly, the idea is that we are using the bounded function $f(\lambda) := \lambda$ on $\sigma(A)$ along with the operator-valued integration described in Proposition 3.4 equation (3). This statement is particularly useful in that it allows us to decompose bounded, linear, self-adjoint operators A and given a meaningful definition of applying to functions to such operators.

Definition 4.2 For a bounded, linear, self-adjoint operator A on $\mathbb{H}$ and a bounded measurable $f : \sigma(A) \to \mathbb{C}$, we can define an operator

$$f(A) = \int_{\sigma(A)} f(\lambda) d\mu^A(\lambda) \tag{8}$$

Considering the Spectral Theorem for compact self-adjoint operators, we can see where this might be useful in regards to Quantum Mechanics.

Theorem 4.3 (Spectral Theorem for Compact Self-Adjoint Operators) Let $T : \mathbb{H} \to \mathbb{H}$ be a compact, self-adjoint operator. Then there exists an orthonormal basis $\{v_\alpha\}_{\alpha \in I}$ for $\mathbb{H}$ such that each v_α is an eigenvector for T. Moreover, for every $x \in \mathbb{H}$,

$$Tx = \sum_{\alpha \in I} \lambda_\alpha (x, v_\alpha) v_\alpha$$

where λ_α is the eigenvalue corresponding to v_α

A proof of this Theorem is covered in [2]. Something to notice though, is the ability to decompose our operator T when it is compact and self-adjoint. Most quantum operators however, are not compact (1) so we needed a stronger theorem to utilize to do the same for operators we care about which can be done for bounded self-adjoint operators with (8). Unfortunately, many quantum operators are also unbounded, requiring an even stronger notion of a 'Functional Calculus' in order to decompose them.

A specific example of this is using Spectral Theory to make sense of the solution to the time-dependent Schrodinger equation. One method of 'solving' the equation is setting $\psi(t) = exp\{-it\hat{H}/\hbar\}\psi_0$. Typically, we might try to define this as a power series expansion of e^x however the Hamiltonian operator $\hat{H}$ is unbounded and will cause such an expansion to diverge. If $\hat{H}$ has true eigenvectors which form an orthonormal basis $\{\psi_k\} \subset \mathbb{H}$, we can define the operator to be one such that

$$e^{-it\hat{H}/\hbar}\psi_k = e^{-it\lambda_k/\hbar}\psi_k$$

As we've seen in the introduction though, not all operators we care about have a set of true eigenvectors. Instead, we use some version of (8) for unbounded operators to get a well-defined alternative to this.

5 References

- [1] *Quantum Theory for Mathematicians* by Brian C. Hall
- [2] *Functional Analysis for the Applied Mathematician* by Todd Arbogast and Jerry L. Bona